Design and Analysis of Algorithms

Mohammad GANJTABESH

Department of Computer Science,
School of Mathematics, Statistics and Computer Science,
University of Tehran,
Tehran, Iran.

mgtabesh@ut.ac.ir



Dept. of Computer Science
University of Tehran

All-Pairs Shortest Paths: Floyd-Warshall algorithm

Design and Analysis of Algorithms
Undergraduate course



Mohammad GANJTABESH
mgtabesh@ut.ac.ir

Dept. of Computer Science
University of Tehran

Graph Theoretical Problems

- Basic definitions
- Graph representation
- Graph traversal (BFS, DFS)
- Topological sort
- Strongly connected components
- Shortest paths
  - Single-source all destination (Dijkestra, Bellman-Ford)
  - All-pairs (Matrix Multiplication, Floyd-Warshall)
- Minimum spanning tree (Kruskal, Prim)

$L^{(m)}$

All-Pairs Shortest Paths: Floyd-Warshall algorithm

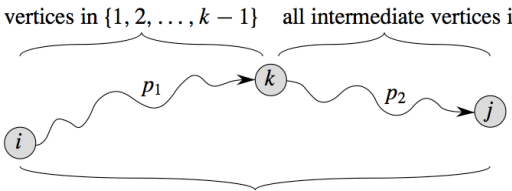
Definition

The intermediate vertex of a simple path $p = \langle v_1, v_2, \cdots, v_l \rangle$ is any vertex of p other than v_1 or v_l, that is, any vertex in the set $\{v_2, v_3, \cdots, v_{l-1}\}$.

- Suppose that $V = \{1, 2, \cdots, n\}$ and $\{1, 2, \cdots, k\} \subset V$.
- For any pair of vertices $i, j \in V$, consider all paths from i to j whose intermediate vertices are all drawn from $\{1, 2, \cdots, k\}$, and let p be a minimum-weight path among them.
  - If k is not an intermediate vertex of path p, then a shortest path from vertex i to vertex j with all intermediate vertices in the set $\{1, 2, \cdots, k-1\}$ is also a shortest path from i to j.
  - If k is an intermediate vertex of path p, then we break down p into $i \overset{p_1}{\rightsquigarrow} k \overset{p_2}{\rightsquigarrow} j$, where p_1 is a shortest path from i to k, p_2 is a shortest path from k to j, and their intermediate vertices are in the set $\{1, 2, \cdots, k-1\}$.

{1, 2, 3, ... k}

All-Pairs Shortest Paths: Floyd-Warshall algorithm



Let $d_{ij}^{(k)}$ be the weight of a shortest path from vertex i to j for which all intermediate vertices are in the set $\{1, 2, \cdots, k\}$.

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0, \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1. \end{cases}$$

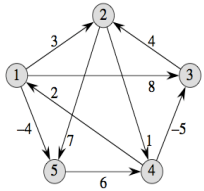
FLOYD-WARSHALL(W)

1 $n \leftarrow rows[W]$
2 $D^{(0)} \leftarrow W$
3 **for** $k \leftarrow 1$ **to** n
4 **do for** $i \leftarrow 1$ **to** n
5 **do for** $j \leftarrow 1$ **to** n
6 **do** $d_{ij}^{(k)} \leftarrow \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$
7 **return** $D^{(n)}$

The total time complexity is $\Theta(n^3)$.

All-Pairs Shortest Paths: Floyd-Warshall algorithm

$D^{(0)} = W$



$$W = D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

کوتاه‌ترین مسیرها وقتی رأس {1} به عنوان رأس میانی بتواند انتخاب شود.

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$