Design and Analysis of Algorithms

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All-Pairs Shortest Paths

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Undergraduate course



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Graph Theoretical Problems

- Basic definitions
- Graph representation
- Graph traversal (BFS, DFS)
- Topological sort
- Strongly connected components
- Shortest paths
  - Single-source all destination (Dijkestra, Bellman-Ford)
  - All-pairs (Matrix Multiplication, Floyd-Warshall)
- Minimum spanning tree (Kruskal, Prim)

All-Pairs Shortest Paths

- Given a weighted directed graph $G = (V, E)$ with a weight function $w : E \mapsto R$.
- The All-Pairs shortest path problem is to find, for every pair of vertices $u, v \in V$, a shortest path from u to v.

The following solutions are available for this problem:

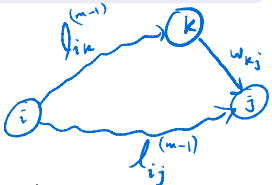
1. Running a single-source shortest-paths algorithm $|V|$ times, once for each vertex as a source.
   - Using Dijkstra's algorithm requires $O(|V|^3)$ time complexity.
   - Using BELLMAN-FORD algorithm requires $O(|V|^2 \times |E|)$ time complexity, which for dense graphs it is $O(|V|^4)$.
2. Using Matrix Multiplication
3. Using FLOYD-WARSHALL algorithm

All-Pairs Shortest Paths: Matrix Multiplication

Let $l_{ij}^{(m)}$ be the minimum weight of any path from vertex i to j that contains at most m edges.

- For $m = 0$:

$$l_{ij}^{(0)} = \begin{cases} 0 & \text{if } i = j, \\ \infty & \text{if } i \neq j. \end{cases}$$



- For $m \geq 1$:

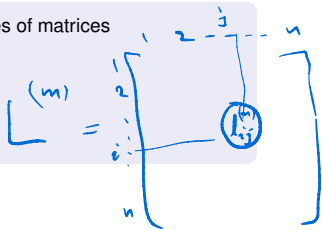
$$\begin{aligned} l_{ij}^{(m)} &= \min\left(l_{ij}^{(m-1)}, \min_{1 \leq k \leq n}\{l_{ik}^{(m-1)} + w_{kj}\}\right) \\ &= \min_{1 \leq k \leq n}\{l_{ik}^{(m-1)} + w_{kj}\} \end{aligned}$$

- Any shortest path from i to j contains at most $n-1$ edges.
- Also, any longer path can not have lower weight, i.e., $\delta(i,j) = l_{ij}^{(n-1)} = l_{ij}^{(n)} = l_{ij}^{(n+1)} = \cdots$.

All-Pairs Shortest Paths: Matrix Multiplication

Let $W = (w_{ij})$ be the weight matrix for graph $G = (V, E)$. We compute a series of matrices $L^{(1)}, L^{(2)}, \cdots, L^{(n-1)}$, where $L^{(m)} = (l_{ij}^{(m)})$ for $m = 1, 2, \cdots, n-1$.

- The final matrix $L^{(n-1)}$ contains the actual shortest-path weights.
- $l_{ij}^{(1)} = w_{ij}$ for all vertices $i, j \in V$, and so $L^{(1)} = W$.



EXTEND-SHORTEST-PATHS(L, W)

```
1  n ← rows[L]
2  let L' = (l'_ij) be an n × n matrix
3  for i ← 1 to n
4       do for j ← 1 to n
5            do l'_ij ← ∞
6               for k ← 1 to n
7                    do l'_ij ← min(l'_ij, l_ik + w_kj)
8  return L'
```

Relation with Matrix Multiplication: $\begin{cases} + \longrightarrow \min \\ \cdot \longrightarrow + \end{cases}$. The time Complexity is $O(n^3)$.

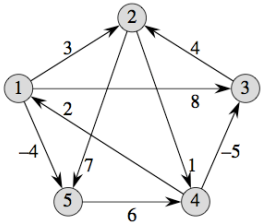
SLOW-ALL-PAIRS-SHORTEST-PATHS(W)

1 $n \leftarrow rows[W]$
2 $L^{(1)} \leftarrow W$
3 **for** $m \leftarrow 2$ **to** $n-1$
4 **do** $L^{(m)} \leftarrow$ EXTEND-SHORTEST-PATHS$(L^{(m-1)}, W)$
5 **return** $L^{(n-1)}$

The total time complexity is $O(n^4)$.

$$\begin{aligned} L^{(1)} &= L^{(0)} \cdot W &&= W\,, \\ L^{(2)} &= L^{(1)} \cdot W &&= W^2\,, \\ L^{(3)} &= L^{(2)} \cdot W &&= W^3\,, \\ &\vdots \\ L^{(n-1)} &= L^{(n-2)} \cdot W &&= W^{n-1} \end{aligned}$$

All-Pairs Shortest Paths: Matrix Multiplication



$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad L^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

$$L^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad L^{(4)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

FASTER-ALL-PAIRS-SHORTEST-PATHS(W)

1 $n \leftarrow rows[W]$
2 $L^{(1)} \leftarrow W$
3 $m \leftarrow 1$
4 **while** $m < n - 1$
5 **do** $L^{(2m)} \leftarrow$ EXTEND-SHORTEST-PATHS$(L^{(m)}, L^{(m)})$
6 $m \leftarrow 2m$
7 **return** $L^{(m)}$

The total time complexity is $O(n^3 \log n)$.

$$
\begin{aligned}
L^{(1)} &= W\,, \\
L^{(2)} &= W^2 &&= W \cdot W\,, \\
L^{(4)} &= W^4 &&= W^2 \cdot W^2 \\
L^{(8)} &= W^8 &&= W^4 \cdot W^4\,, \\
&\vdots \\
L^{(2^{\lceil \lg(n-1) \rceil})} &= W^{2^{\lceil \lg(n-1) \rceil}} &&= W^{2^{\lceil \lg(n-1) \rceil - 1}} \cdot W^{2^{\lceil \lg(n-1) \rceil - 1}}
\end{aligned}
$$